ON A NONLINEAR PROBLEM OF THE WAVE EQUATION IN ONE-DIMENSIONAL SPACE

(OB ODNOI NELINEINOI ZADACHE DLIA VOLNOVOGO URAVNENIIA V ODNOMERNOM PROSTRANSTVE)

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Iu.P. SAMARIN

(Kuibyshev)

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The construction of the solution to a nonlinear problem of mathematical physics is given; a linear problem of similar type has been considered in [1,2].

Let a string be stretched along the x-axis and have as one of its boundaries a weight; the string passes through it and the weight can slide freely along the x-axis under the action of the pressure from the oscillating part of the string. The equations of motion of the weight and of oscillation of the string must be determined jointly; therefore this is a nonlinear problem with a moving boundary. For the solution of this problem we can take advantage of its transformation to a Cauchy problem, as is carried out below.

If the weight begins its motion from the point $x = 0$ and the string moves it in the direction of increasing x, then for the oscillating portion of the string we have

$$u|_{t=0} = \varphi(x), \qquad x \leqslant 0, \qquad \varphi(0) = 0$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \left[\frac{\partial u}{\partial t}\right]_{t=0} = \psi(x), \qquad x \leqslant 0, \qquad \psi(0) = 0 \tag{1}$$

$$u|_{x=X(t)} = 0, \qquad X(0) = 0 \tag{2}$$

where $x = X(t)$ is the equation of motion of the weight.

The solution of this Cauchy problem has the form

$$u = \frac{1}{2}\varphi(x-t) + \frac{1}{2}\varphi(x+t) + \frac{1}{2}\int_{x-t}^{x+t}\psi(\eta)\,d\eta \tag{3}$$

If we set $\psi(x) \equiv 0$ for $x > 0$, then from (3) and (2) we have

$$\varphi(\tau) = -\varphi(\tau - 2\chi(\tau)) + \int_0^{\tau-2\chi(\tau)}\psi(\eta)\,d\eta, \qquad \tau = X(t) + t, \quad \chi(\tau) = t \tag{4}$$

This formula gives the required continuation of the function φ to positive values of the argument ($\tau > 0$, since $-t < X(t) < t$), and therefore with the use of (3) for $x + t > 0$ we can write

$$u = \frac{1}{2}\varphi(x-t) - \frac{1}{2}\varphi(\xi) + \frac{1}{2}\int_{x-t}^{\xi}\psi(\eta)\,d\eta \qquad (\xi = x + t - 2\chi(x+t)) \tag{5}$$

We now make use of the equation $mX^{\cdot\cdot} = F$, where m is the mass of the weight and F is the force exerted by the string on the weight. From energetical considerations it can be established that

$$F(t) = \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2_{x=X(t)}(1 - X^{\cdot 2}) \tag{6}$$

where T is the tension in the string. Now, using the above equation for the right-hand part of (6), we get the equation for the determination of the weight motion

$$X^{\cdot\cdot} = k^2\,[\varphi'(X - t) - \psi(X - t)]^2\,\frac{1 - X^{\cdot}}{1 + X^{\cdot}} \qquad \left(k^2 = \frac{T}{2m}\right) \tag{7}$$

In a particular case, if

$$X(0) = X^{\cdot}(0) = 0, \qquad \psi(x) \equiv 0, \qquad |\varphi'(x)| = a = \text{const}$$

where $\varphi(x)$ is the "saw-tooth" function for $x < 0$, the solution of Equation (7) has the form .

$$k_1^2 t = 1 - \sqrt{1 + 2k_1^2(t - X)} - 2\ln\left[2 - \sqrt{1 + 2k_1^2(t - X)}\right] \qquad (k_1 = ka) \tag{8}$$

The curve (8) has as asymptotic characteristic

$$t = x + \frac{3}{2k_1^2}$$

and its form resembles a hyperbola. This simple particular case demonstrates the character of the motion of a weight due to an oscillating string.

In the general case Equation (7) has the integral

$$\int_0^{X-t} \frac{dy}{-2 + \sqrt{C_1 - 2k^2\Phi(y)}} = t + C_2 \qquad \left(\Phi(y) = \int_0^y [\varphi'(\eta) - \psi(\eta)]^2\,d\eta\right)$$

Constants C_1 and C_2 are selected according to the values $X(0)$ and $X^{\cdot}(0)$.

In conclusion we note that the use of the method described assumes the knowledge of the function $\varphi(x)$ for all $x \leqslant 0$, which is the case for a semi-infinite string; for a finite string the solution can be obtained by the successive continuation of the function $\varphi(x)$ first past the right, then past the left boundaries of the string.

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